# Overcoming near-degeneracy in the autologistic actor attribute model

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#### Abstract

The autologistic actor attribute model, or ALAAM, is the social influence counterpart of the better-known exponential-family random graph model (ERGM) for social selection. Extensive experience with ERGMs has shown that the problem of near-degeneracy which often occurs with simple models can be overcome by using "geometrically weighted" or "alternating" statistics. In the much more limited empirical applications of ALAAMs to date, the problem of near-degeneracy, although theoretically expected, appears to have been less of an issue. In this work I present a comprehensive survey of ALAAM applications, showing that this model has to date only been used with relatively small networks, in which near-degeneracy does not appear to be a problem. I show near-degeneracy does occur in simple ALAAM models of larger empirical networks, define some geometrically weighted ALAAM statistics analogous to those for ERGM, and demonstrate that models with these statistics.

Keywords- autologistic actor attribute model, ALAAM, exponential-family random graph model, ERGM, near-degeneracy

## 1 Introduction

The autologistic actor attribute model (ALAAM) is a statistical model of social influence, or contagion on a social network. The ALAAM, first introduced by Robins et al. (2001) and extended by Daraganova (2009) to its current form, is a variant of the exponential-family random graph model (ERGM), a widely-used model for social networks (Lusher et al., 2013; Ghafouri and Khasteh, 2020). Both ALAAM and ERGM are models for cross-sectional data, that is, a network and nodal attributes observed at one point in time (or preferably, for the ALAAM, the network and nodal attributes at one point, and the outcome binary attribute at a suitable later point (Parker et al., 2022)). The distinction between the ERGM and the ALAAM is that the ERGM models the probability of network ties, conditional on nodal attributes, while the ALAAM models the probability of a (binary) nodal attribute, conditional on the network (and other nodal attributes).

The ALAAM, modeling the probability of attribute *Y* (a vector of binary attributes) given the network *X* (a matrix of binary tie variables) can be expressed as (Daraganova and Robins, 2013):

$$\Pr(Y = y | X = x) = \frac{1}{\kappa(\theta_I)} \exp\left(\sum_I \theta_I z_I(y, x, w)\right)$$
(1)

where  $\theta_I$  is the parameter corresponding to the network-attribute statistic  $z_I$ , in which the "configuration" I is defined by a combination of dependent (outcome) attribute variables y, network variables x, and actor covariates w, and  $\kappa(\theta_I)$ is a normalizing quantity which ensures a proper probability distribution. Table 1 shows some simple configurations for undirected networks used in this work, while Table 2 shows a more extensive list of configurations for directed networks used in this work.

Both ERGMs and ALAAMs, because of the presence of the intractable normalizing constant,  $\kappa(\theta_I)$  in (1), usually require Markov chain Monte Carlo (MCMC) methods for maximum likelihood estimation (MLE) of the parameters (Snijders, 2002; Hunter and Handcock, 2006; Hunter et al., 2012; Lusher et al., 2013; Amati et al., 2018; Koskinen, 2020). Once the parameters and their standard errors are estimated, they can be used for inferences regarding the

#### Table 1: Configurations used in ALAAMs for undirected networks in this work.

| Name          | Illustration | Description  |
|---------------|--------------|--|
| Density       | $\bigcirc$   | Baseline attribute density (incidence). Also used with directed networks   |
| Activity      |              | Tendency for actor with the attribute to have ties   |
| Contagion     |              | Tendency for actor with the attribute to be tied to an actor also with the at-<br>tribute  |
| attribute_oOc | 0            | Covariate effect for continuous covariate <i>attribute</i> . The "_oOc" notation is from IPNet (Wang et al., 2009a), and we may omit this when there is no ambiguity, e.g. "Age_oOc" may also be written simply as "Age". Also used in directed networks |

Legend:

Node with outcome attribute

O Node irrespective of outcome attribute

corresponding configurations. A parameter estimate that is statistically significant and positive indicates an overrepresentation of the corresponding configuration, conditional on all the other parameters in the model. Conversely, a parameter that is statistically significant and negative indicates an under-representation of that configuration given all the others in the model.

A well-known problem with ERGMs is that simple model specifications can lead to "near-degeneracy" in which the MLE does not exist, or the model generates distributions of graphs in which most of the probability mass is placed on (nearly) empty or (nearly) complete graphs (Handcock, 2003; Snijders et al., 2006; Hunter, 2007; Schweinberger, 2011; Chatterjee and Diaconis, 2013; Schweinberger et al., 2020). This problem is usually overcome by the use of more complex "alternating" or "geometrically weighted" configurations (Snijders et al., 2006; Robins et al., 2007; Hunter, 2007; Lusher et al., 2013), however other forms of additional mathematical structure can also be used to solve (or avoid) the problem of near-degeneracy (Schweinberger et al., 2020)

Since the ALAAM, like the ERGM, is a type of Gibbs random field, and specifically the ALAAM derives from the autologistic Ising model (Besag, 1972), it is to be expected, that, like the ERGM, problems of near-degeneracy would arise due to the well-known phase transition behaviour in such models (Fellows and Handcock, 2017; Stoehr, 2017). It has, however, been observed that for ALAAMs "this is less of an issue" (Koskinen and Daraganova, 2022, p.1856), and indeed "alternating" or "geometrically weighted" statistics have to date not been described for ALAAMs, with published models using simple configurations such as those shown in Table 1 and Table 2.

In this work I will show that this could be due to the somewhat limited experience with ALAAMs to date, and specifically that their use has been restricted to relatively small networks. I demonstrate that near-degeneracy does occur in ALAAMs with empirical networks, and propose new geometrically weighted statistics, analogous to the geometrically weighted degree statistics for ERGMs, that overcome this problem and allow estimation of ALAAM models that could not be estimated using, for undirected networks, the Activity statistic (Table 1) or, for directed networks, the Sender and Receiver statistics (Table 2).

### 2 Survey of ALAAM applications

As noted by Parker et al. (2022, p. 517), empirical experience with ALAAMs is recent and limited. This is particularly so relative to the social selection model ERGM, which is widely used across a variety of domains; for a recent survey see Ghafouri and Khasteh (2020), as well as, for example Lusher et al. (2013); Amati et al. (2018); Cimini et al. (2019). It is therefore practical to present a comprehensive survey of empirical ALAAM usage. I used Google Scholar to search for "autologistic actor attribute model" (search date 24 August 2023), which resulted in 34 hits. Note that, as is well known, Google Scholar includes not just peer-reviewed publications, but "grey literature" such as PhD theses, unpublished preprints and technical reports, among others. I chose not to restrict this literature survey to peer-reviewed publications, but to also include preprints, conference presentations, and PhD theses, as long as they

# Table 2: Configurations used in ALAAMs for directed networks in this work.

| Name                   | Illustration | Description  |
|------------------------|--------------|--|
| Sender                 |              | Tendency of actors with the attribute to have outgoing ties (activity)   |
| Receiver               |              | Tendency of actors with the attribute to have incoming ties (popularity)   |
| Contagion              |              | Tendency of the attribute to be present in both actors connected by directed tie   |
| Reciprocity            |              | Tendency of the attribute to be present in an actor connected to another by a reciprocated (mutual) tie  |
| Contagion reciprocity  |              | Also known as mutual contagion. Tendency of the attribute to be present in both actors connected by a reciprocated tie                           |
| Ego in-two-star        |              | Tendency of the attribute to be present in an actor with additional incoming ties over Receiver  |
| Ego out-two-star       |              | Tendency of the attribute to be present in an actor with additional outgoing ties over Sender  |
| Mixed-two-star         |              | Tendency of the attribute to be present in an actor in the broker position be-<br>tween two other nodes (local brokerage)                        |
| Mixed-two-star source  |              | Tendency of the attribute to be present in an actor in the source position in local brokerage  |
| Mixed-two-star sink    |              | Tendency of the attribute to be present in an actor in the sink position in local brokerage  |
| Transitive triangle T1 |              | Tendency of the attribute to be present in an actor in a transitive triangle, the broker position in Mixed-two-star bypassed by a transitive tie |
| Transitive triangle T3 |              | Contagion clustering: tendency of the attribute to be present in all three actors in a transitive triangle                                       |

met the same criteria I defined for publications, namely:

- 1. The ALAAM model is applied to empirical data. This excludes, for example, Stivala et al. (2020b), which is a simulation study, rather than an application to empirical data.
- 2. The model used was an ALAAM as described in this work; the family of model implemented for example by IPNet (Wang et al., 2009a) and its successor software MPNet (Wang et al., 2014, 2022). Note that this excludes the original ALAAM paper (Robins et al., 2001), in which the outcome variable is not dichotomous (binary), but rather polytomous (three values). This paper also predates the introduction of the name "autologistic actor attribute model", and uses maximum pseudo-likelihood for estimation. This criterion also excludes the more recent exponential-family random network model (ERNM), a generalization of the ERGM and ALAAM, which models both social selection and social influence simultaneously (Fellows and Handcock, 2012, 2013; Wang et al., 2023).
- 3. The work is either publicly available, or available to me via my affiliation at Università della Svizzera italiana.

This initial search was supplemented by searching for the same terms using Clarivate Web of Science and Elsevier Scopus (search date 30 August 2023). These searches results in 7 and 34 results, respectively, with a large overlap with the Google Scholar results. I further supplemented these results by adding some works with which I was personally familiar, because, for example, I am an author or I was informed of their existence by an author. The final list of 19 works, containing 25 empirical ALAAM models, is shown in Table 3.

| Citation                        | Network description   | Outcome description  | Network | Estimation | Comments   |
|---------------------------------|---|--|---------|------------|--|
|                                 |   |  | size    | method     |  |
| Barnes et al.<br>(2020)         | Multilevel social-ecological: households<br>with communication relationships, fish<br>species with trophic relationships, cross-<br>level fishing targets   | Two models: Adaptive action and transformative action  | 198     | MPNet      | Multilevel network with 138 households, 60 fish species  |
| Bodin and Chen (2023)           | Multilevel social-ecological: affective re-<br>lations, organization-based collaboration,<br>rangeland use, and species dispersal and<br>livestock movement   | Highly adaptive (dichotomized<br>from continuous measure of<br>change in number of grazing<br>patches) | ?       | MPNet      | Network size not specified, but<br>from figures in S.I. appears to be<br>less than 100   |
| Bryant et al. (2017)            | Social network in a post-disaster commu-<br>nity  | Two models: Probably depres-<br>sion and probably posttraumatic<br>stress disorder (PTSD)              | 558     | MPNet      | Directed network   |
| Daraganova and<br>Robins (2013) | Social network in a high unemployment region  | Unemployment   | 551     | IPNet      | Two-wave snowball sample. In-<br>cludes geographic proximity co-<br>variate  |
| Diviák et al.<br>(2020)         | Collaboration network among organized crime offenders   | Female gender  | 1390    | IPNet      | Not being used as a social influ-<br>ence model, rather a network dis-<br>criminant analysis. Includes pre-<br>existing ties network as a setting<br>network covariate |
| Fujimoto et al.<br>(2019)       | Multilevel referral-affiliation network of<br>client-referral ties from community-based<br>organizations (CBOs) to PrEP providers<br>and utilization by young men who have<br>sex with men (YMSM) of CBOs and PrEP<br>providers | Pre-exposure prophylactic<br>(PrEP) uptake   | 284     | MPNet      | Houston (25 venues and 259 YMSM)   |
|                                 |   |  | 308     |            | Chicago (24 venues and 284 YMSM)   |
| Gallagher<br>(2019)             | Core discussion network among English-<br>for-Academic-Purposes international stu-<br>dents   | Willingness to communicate in<br>English (dichotomized from per-<br>centage of time)                   | 67      | MPNet      | Directed network   |
| Kashima et al. (2013)           | Social network in a regional community  | Perceived descriptive norm of<br>high community engagement<br>(dichotomized from continuous<br>scale)  | 104     | IPNet      | Two-wave snowball sample   |

Table 3: Literature survey of works using the ALAAM.

| Citation      | Network description                       | Outcome description               | Network | Estimation | Comments  |
|---------------|---|-----------------------------------|---------|------------|---|
|               |   |                                   | size    | method     |   |
| Koskinen and  | Directed friendship network in an all-    | High masculinity index (di-       | 106     | R code     | Bayesian inference with missing                       |
| Daraganova    | male school                               | chotomized from Masculine         |         |            | data. Also includes re-analysis                       |
| (2022)        |   | Attitudes Index)                  |         |            | of the Daraganova and Robins (2013) unemployment data |
|               | Directed friendship network from Stock-   | Intention to proceed to higher    | 403     |            | (2013) unemployment data                              |
|               | holm Birth Cohort data                    | secondary education               |         |            |   |
| Letina (2016) | Co-authorship network for two fields of   | High productivity (two models:    | 125     | MPNet      | Psychology  |
|               | social science in Croatia                 | dichotomized from number of       |         |            |   |
|               |   | publications, or H-index)         | 100     |            | 0 1   |
| Lating at al  | Co authorship network for three fields of | One or more ties outside the na   | 102     | MDNat      | Sociology<br>Psychology                               |
| (2016)        | social science in Croatia                 | tional and/or disciplinary com-   | 100     | WII Net    | rsychology  |
| ()            |   | munity (NDC)                      |         |            |   |
|               |   | • • •                             | 136     |            | Sociology   |
|               |   |                                   | 250     |            | Educational sciences                                  |
| Matous and    | Advice network regarding cocoa farming    | Farmers' use of fertilizer        | 71      | MPNet      | Undirected network. Fourteen                          |
| Bodin (2021)  | practices                                 |                                   |         |            | (mean 71)   |
| Neidhardt     | Friendship network of schoolchildren in   | Smoking behaviour (di-            | 160     | IPNet      | Undirected network                                    |
| (2016)        | Glasgow                                   | chotomized from occasionally or   |         |            |   |
|               |   | regularly)                        |         |            |   |
|               | Partners (co-players) in an online game   | Cancelled subscription to game    | 2587    |            | Took two days to estimate in IP-                      |
|               |   |                                   |         |            | Net and the results are not stable                    |
|               |   |                                   |         |            | (Neiuliaiut, 2010, p. 100)                            |
| Ocelik et al. | Long-term cooperation network of people   | High-level participation (di-     | 38      | MPNet      | Undirected network                                    |
| (2021)        | opposed to the rescinding of coal-mining  | chotomized from continuous        |         |            |   |
|               | limits in the Czech Republic              | differential participation scale) |         |            |   |
| Parker et al. | Directed advice network among students    | Two models: high perfor-          | 133     | MPNet      |   |
| (2022)        | in a management course                    | (dichotomized from grades)        |         |            |   |
| Rank (2014)   | Collaboration network among top man-      | Firm survival                     | 53      | IPNet      | Undirected network. Paper refers                      |
| × /           | agers of all member companies and orga-   |                                   |         |            | to the ALAAM model as ERGM                            |
|               | nizations in a regional biotech network   |                                   |         |            | for social influence                                  |

Table 3: Literature survey of works using the ALAAM.

| Citation         | Network description                     | Outcome description            | Network | Estimation | Comments                          |
|------------------|---|--------------------------------|---------|------------|-----------------------------------|
|                  |   |                                | size    | method     |                                   |
| Song et al.      | Social network of an online weight-loss | Self-monitoring performance    | 724     | IPNet      | Undirected network. Estimation    |
| (2020)           | community                               | (dichotomized from continuous  |         |            | method not reported, but effect   |
|                  |   | score)                         |         |            | names indicate IPNet              |
| Stadtfeld et al. | Positive interactions, friendship, and  | Passing the final exam         | 163     | MPNet      | Analysis uses stochastic actor-   |
| (2019)           | studying together networks among engi-  |                                |         |            | oriented model (SAOM) (Sni-       |
|                  | neering undergraduate students          |                                |         |            | jders, 2017) for network evolu-   |
|                  |   |                                |         |            | tion, with ERGM for robustness    |
|                  |   |                                |         |            | check, and linear regression for  |
|                  |   |                                |         |            | final exam result, with logistic  |
|                  |   |                                |         |            | regression, network autocorrela-  |
|                  |   |                                |         |            | tion, and ALAAM as robustness     |
| Stivele et el    | Dinacton interlects network             | Esmala conden                  | 12059   |            | As in Divisit at al. (2020) not   |
| (2022h)          | Director interiock network              | remaie gender                  | 12038   | ALAAMEE    | As in Diviak et al. (2020), not   |
| (20230)          |   |                                |         |            | model rather a network discrim    |
|                  |   |                                |         |            | inouel, famer a network discrimi- |
|                  |   |                                |         |            | 9971 directors and 2087 compa-    |
|                  |   |                                |         |            | nies Estimated with stochastic    |
|                  |   |                                |         |            | approximation                     |
| Wood (2019)      | Friendship network in a novel mobile    | Commitment to vote in an elec- | 74      | MPNet      | Undirected network                |
|                  | platform                                | tion                           | ,.      |            |                                   |

| Table 3: Literature survey of works using the ALAAM. |  |
|--|--|
|--|--|

In all but two cases, the ALAAM was estimated with stochastic approximation (Snijders, 2002), using either the IPNet or MPNet software. The first exception is Koskinen and Daraganova (2022), which describes Bayesian estimation of the ALAAM, accompanied by R code which implements this method. The second exception is Stivala et al. (2023b), in which the ALAAM is estimated using the ALAAMEE software (Stivala et al., 2023a), also used in this work. In Stivala et al. (2023b), ALAAM models for the 12058 node bipartite director interlock network were estimated using stochastic approximation (the same algorithm implemented in IPNet and MPNet). However a converged ALAAM for the larger director interlock network (Evtushenko and Gastner, 2020) with 356638 nodes (321869 directors and 34769 companies) could not be found, using either the stochastic approximation or equilibrium expectation algorithms implemented in ALAAMEE. In contrast, converged ERGM models for both networks, using "alternating" star statistics for bipartite networks (Wang et al., 2009b) were found, using the EstimNetDirected software (Stivala et al., 2020a)

The mean network size (number of nodes) in Table 3 is 832.1, the median is 160, and the maximum is 12058. (Of the 26 models, one did not specify the network size, and hence these results are over 25 networks.) However, excluding the single use of ALAAMEE, the mean is 364.4, the median 160, and the maximum 2587. Even for this 2587 node network, it is noted that the estimation using IPNet took two days, and the results were "not stable" (Neidhardt, 2016, p. 106). The largest network for which estimation (with IPNet) was not problematic is the 1390 node network in Diviák et al. (2020).

The largest network used in the simulation studies described in Stivala et al. (2020b) is 4430 nodes, however although this is an empirical network, the binary outcome attribute is not itself an empirical covariate, but rather simulated from an ALAAM model for the purposes of testing statistical inference using a model with known parameters.

This demonstrates that, with the exception of some very recent (and currently ongoing) work (Stivala et al., 2023a,b), empirical experience with ALAAMs is mostly restricted to networks of the order of a few hundred nodes in size, and certainly no larger than a few thousand.

### **3** Near-degeneracy with standard ALAAM parameters

In this and the following sections, three networks will be used as examples. First, a network of friendship relations between students in a high school in Marseilles, France, collected in December 2013 by the SocioPatterns research collaboration (Mastrandrea et al., 2015). This is a directed network of friendship relations, where an arc from a node *i* to a node *j* indicates that student *i* reported a friendship with student *j*. The school class and gender (male or female) of each student is known (one is unknown), and male gender is used as the binary "outcome" attribute. In this way, the ALAAM is not being used as a social influence model (it is not assumed that gender is affected by network position), but rather as a way of making inferences about the structural positions of males in the network, as was done for female gender in Diviák et al. (2020); Stivala et al. (2023b). Similar considerations apply to the other two networks: I am not actually using ALAAM as a social influence model, but merely using these examples to illustrate problems of near-degeneracy and how to overcome it with the new geometrically weighted activity statistic.

The second network is a large online social network of GitHub (an online platform for software development) software developers, collected in June 2019 (Rozemberczki et al., 2021). Nodes are developers (who have "starred" at least ten repositories) and undirected edges are mutual "follower" relationships between them. This data set was created for binary node classification, and the target binary feature, which is used here as the binary outcome attribute, is the developer type, either "web" or "machine learning" (Rozemberczki et al., 2021). Here this developer type is used as the outcome variable — it is not clear which developer type the nonzero value of this variable indicates, so I do not ascribe any meaning to ALAAM inferences regarding this variable (and, again, nor do I actually make the assumption that the developer type is subject to social influence).

The third network is the "Pokec" online social network, at one time the most popular such network in Slovakia (Takac and Zabovsky, 2012). Arcs in this network represent directed "friendship" relations, and the nodes are annotated with a number of attributes, including age and gender. Again, male gender is used as the binary "outcome" attribute here. As described Stivala et al. (2020a), the 20 "hub" nodes with degree greater than 1000 are removed. Two versions of this network are considered here, the original directed version, and an undirected version in which only mutual "friendship" relations are retained, as is done in Kleineberg and Boguñá (2014).

Descriptive statistics of the networks are shown in Table 4, and of the nodes with  $(y_i = 1)$  and without  $(y_i = 0)$  the outcome attribute in Table 5. The high school network is of a size that is typical of current publications using the ALAAM (see Section 2), but the GitHub and Pokec networks are orders of magnitude larger. These are too large

to estimate in practical time using the stochastic approximation algorithm, and so although the high school network models will be estimated using stochastic approximation, the GitHub and Pokec models will be estimated using the equilibrium expectation algorithm instead, which is suitable for very large networks (Byshkin et al., 2016, 2018; Borisenko et al., 2020; Stivala et al., 2020a)

| Network     | Directed | Nodes   | Size of giant | Mean   | Max.      | Max.       | Density | Clustering  |
|-------------|----------|---------|---------------|--------|-----------|------------|---------|-------------|
|             |          |         | component     | degree | in-degree | out-degree |         | coefficient |
| GitHub      | No       | 37700   | 37700         | 15.33  | 9458      | 9458       | 0.00041 | 0.01236     |
| Pokec       | No       | 1632783 | 1197779       | 10.16  | 671       | 671        | 0.00001 | 0.06854     |
| Pokec       | Yes      | 1632783 | 1632199       | 18.69  | 949       | 998        | 0.00001 | 0.05369     |
| High school | Yes      | 134     | 128           | 4.99   | 15        | 16         | 0.03748 | 0.47540     |

Table 4: Network descriptive statistics for the example networks.

Network statistics computed using the igraph (Csárdi and Nepusz, 2006) R package. "Clustering coefficient" is the global clustering coefficient (transitivity)

| Network     | Directed | $y_i = 1$ |           |                 |           |                 |
|-------------|----------|-----------|-----------|-----------------|-----------|-----------------|
|             |          | nodes %   | Outcome y | $w_i = 0$ nodes | Outcome y | $v_i = 1$ nodes |
|             |          |           | Mean      | Mean            | Mean      | Mean            |
|             |          |           | in-degree | out-degree      | in-degree | out-degree      |
| GitHub      | No       | 26        | 17.67     | 17.67           | 8.63      | 8.63            |
| Pokec       | No       | 49        | 10.68     | 10.68           | 9.62      | 9.62            |
| Pokec       | Yes      | 49        | 20.55     | 18.34           | 16.78     | 19.06           |
| High school | Yes      | 40        | 4.79      | 4.69            | 5.28      | 5.43            |

Table 5: Mean degrees of nodes with and without the outcome attribute.

The motivation for this work was my inability to find converged (non-degenerate) ALAAM models for large networks, such as the Pokec and GitHub networks, when the Activity parameter was included, as it typically is in an ALAAM model. Figure 1 shows why this is so. These plots show, for the (undirected) GitHub and Pokec networks, the value of the Activity statistic in simulated ALAAM outcome vectors, as the corresponding parameter is varied from -1.0 to 1.0 in increments of 0.01. Each data point is the result of one of 100 samples from the ALAAM distribution drawn every  $10^6$  iterations after a burn-in period of  $10^7$  iterations, using the simulateALAAM function of ALAAMEE (Stivala et al., 2023a). The Density and Contagion parameters are fixed at -0.50 and 0.50, respectively, for GitHub, and -0.155 and -0.008, respectively, for Pokec. These values were chosen to be in the vicinity of the estimated values in the (non-converged) models. It is clear that there is a near discontinuity in the Activity statistic, with a strong peak in its variance, characteristic of the phase transition in the Ising and Potts models (Stoehr, 2017). This is similar to the well-known near-degeneracy in Markov (for example, edge-star and edge-triangle) ERGM models, as described in, for example, Handcock (2003); Snijders et al. (2006); Robins et al. (2007); Koskinen and Daraganova (2013), which often prevents the estimation of such models.

## **4** A geometrically weighted activity statistic

Since this near-degeneracy in the ALAAM with the Activity parameter appears very similar to that which occurs in the ERGM with the star parameter, the solution may well also be similar. In the ERGM, near-degeneracy in such models is usually avoided by using, rather than two-star, three-star, etc. terms, an "alternating *k*-star" or "geometrically weighted degree" parameter (Robins et al., 2007; Lusher et al., 2013), as proposed by Snijders et al. (2006); Hunter (2007).

Here I will follow Snijders et al. (2006, s. 3.1.1) in using geometrically weighted degree counts for ERGMs, in order to create a geometrically weighted activity statistic for ALAAMs.

First, note that the Activity statistic is

$$z_{\text{Activity}}(y) = \sum_{i:y_i=1}^{\infty} d(i), \tag{2}$$



Figure 1: Effect on the Activity statistic (scatterplot, left, and variance, right) of varying the Activity parameter, in the GitHub social network (top) and undirected Pokec social network (bottom). The red dashed horizontal line shows the observed value of the statistic.

where d(i) denotes the degree of node *i*. That is, it is the sum of the degrees of each node for which the outcome binary attribute  $y_i = 1$ . And hence the change statistic (Hunter and Handcock, 2006; Snijders et al., 2006; Hunter et al., 2012), that is, the change in the statistic when  $y_i$  is changed from 0 to 1 for some node *i*, for the Activity statistic, is just d(i).

The geometrically weighted degree count for ERGM is defined by Snijders et al. (2006, (Eq. 11)) as

$$u_{\alpha}^{(d)}(x) = \sum_{k=0}^{N-1} e^{-\alpha k} d_k(x) = \sum_{i=1}^{N} e^{-\alpha d(i)},$$
(3)

where *N* is the number of nodes,  $d_k(x)$  is the number of nodes of degree *k*, and  $\alpha > 0$  is the degree weighting parameter, controlling the geometric rate of decrease of weights as node degree increases (Snijders et al., 2006, p. 112). Analogously, I define the geometrically weighted activity (GWActivity) statistic for ALAAMs as

$$z_{\text{GWActivity}(\alpha)}(y) = \sum_{i:y_i=1} e^{-\alpha d(i)}.$$
(4)

The change statistic for GWActivity is then simply

$$\delta_{\text{GWActivity}(\alpha)}^{(i)}(y) = e^{-\alpha d(i)}.$$
(5)

Note that  $\alpha$  is not a model parameter, but rather is fixed at a given value (although of course it may be adjusted as necessary for better convergence or model fit). For large values of  $\alpha$ , the contribution of higher degree nodes with the outcome attribute is decreased. As  $\alpha$  decreases to zero, increasing weight is placed on ALAAM outcome vectors with the outcome attribute on high degree nodes.

If  $\alpha$ , or an equivalent parameter, is estimated as part of the model, then the model becomes a member of the curved exponential family (Hunter, 2007). However in this work the value of  $\alpha$  is fixed at the "traditional" value of  $\alpha = \ln(2)$  as in Snijders et al. (2006). Via the mathematical relationships described in Snijders et al. (2006); Hunter (2007), this corresponds to the default value of the decay parameter  $\lambda = 2$  for the alternating *k*-star parameter (Robins et al., 2007) familiar to users of the PNet and MPNet software.

As described in Snijders et al. (2006, p. 114), the ERGM change statistic corresponding to the geometrically weighted degree statistic (3) is a non-decreasing function, with the change becoming smaller as the degrees become larger, and for  $\alpha > 0$  the change statistic is negative. Hence the conditional log-odds of a tie is greater for a tie between high degree nodes than for a tie between low degree nodes.

The ALAAM change statistic for GWActivity (5), by contrast, is positive, and a non-increasing function, when  $\alpha > 0$ . Changing a node outcome attribute from zero to one causes the GWActivity statistic (4) to increase, but by a larger amount for low degree nodes than high degree nodes. Hence the conditional log-odds for a node having the outcome attribute is greater for a low degree node than for a high degree node, but in a non-linear fashion, with the marginal decrease in log-odds decreasing geometrically with degree.

Note that the geometrically weighted activity statistic for ALAAMs I have defined here is analogous to the that for ERGMs defined by Snijders et al. (2006), and not the different geometrically weighted degree statistic defined by Hunter (2007), and familiar to users of the statnet ERGM software packages (Handcock et al., 2008, 2016, 2022; Krivitsky et al., 2023). The relationship between those statistics is discussed Hunter (2007, p. 222).

For directed networks, I also define GWS ender, the geometrically weighted sender statistic, as

$$z_{\text{GWSender}(\alpha)}(y) = \sum_{i:y_i=1} \exp\left(-\alpha d^{(\text{out})}(i)\right),\tag{6}$$

where  $d^{(\text{out})}(i)$  is the out-degree of node *i*. GWReceiver, the geometrically weighted receiver statistic is

$$z_{\text{GWReceiver}(\alpha)}(y) = \sum_{i:y_i=1} \exp\left(-\alpha d^{(\text{in})}(i)\right),\tag{7}$$

where  $d^{(in)}(i)$  is the in-degree of node *i*. The corresponding change statistics are

$$\delta_{\text{GWSender}(\alpha)}^{(i)}(y) = \exp\left(-\alpha d^{(\text{out})}(i)\right)$$
(8)



Figure 2: Scatterplots of the effect on the Geometrically Weighted Activity statistic of varying the Geometrically Weighted Activity parameter in the GitHub social network (left) and undirected Pokec social network (right). The red dashed horizontal line shows the observed value of the statistic.

and

$$\delta_{\text{GWReceiver}(\alpha)}^{(i)}(y) = \exp\left(-\alpha d^{(\text{in})}(i)\right). \tag{9}$$

In order to examine the behaviour of the new GWActivity statistic to verify that it removes the near-degenerate behaviour apparent with the standard Activity statistic, I conducted simulation experiments similar to those described above for Figure 1. Figure 2 shows, for the same two networks, the value of the GWActivity statistic as the corresponding parameter is varied (again in increments of 0.01, and with the same burn-in and iterations). The Density and Contagion parameters are fixed at -1.28 and 0.002 for GitHub, and the same as described for Figure 1 for Pokec. These parameters were chosen to be in the vicinity of estimated parameters (from models similar to those described in Section 5.2).

Figure 2 shows that the phase transition apparent in Figure 1 no longer occurs with this parameterization, with the statistic instead being a smoothly non-decreasing function of the parameter. Furthermore, the curve of the statistic values intersects with the observed value at a point where the slope of curve is not extreme, and there is no near discontinuity (unlike Figure 1), suggesting that maximum likelihood estimation is less likely to be problematic.

#### 4.1 Interpretation of the new parameters

As described in Daraganova and Robins (2013), the interpretation of the Activity parameter is that, if it is positive, it means that an actor with multiple ties is more likely to have the outcome attribute. The two-star and three-star parameters then allow for nonlinear dependence on the number of ties. Interpretation of the GWActivity parameter, however, is not quite so straightforward.

Snijders et al. (2006), in the context of the ERGM, describes how the geometrically weighted degree statistic can be re-written in terms of the numbers of *k*-stars, where the weights on the *k*-stars have alternating signs, so that the positive weights of some are balanced by the negative weights of the others. In this way, the single alternating *k*-star parameter replaces a whole series of two-star, three-star, etc. parameters, which when estimated from empirical networks, tend to have parameters with alternating signs (Koskinen and Daraganova, 2013). The interpretation of the alternating *k*-star in ERGM, then, is in terms of the the degree distribution: a positive parameter a relatively more equal degree distribution (Robins et al., 2007; Koskinen and Daraganova, 2013). Confusingly (Levy et al., 2016; Martin, 2020; Stivala, 2020b), the interpretation of the statnet gwdegree parameter defined in Hunter (2007) has the opposite interpretation regarding the sign: a negative gwdegree parameter indicates centralization of edges, and a positive gwdegree parameter indicates dispersion of edges (Levy, 2016; Levy et al., 2016).

In the present context, that of the ALAAM, however, the degree distribution is not being modeled, as the network is fixed. Instead, the binary outcome vector is being modeled. Therefore it is not useful to examine the effect of a parameter on the degree distribution of the whole network, but rather of the degree distribution of those nodes which have the outcome attribute (nodes *i* such that  $y_i = 1$ ). As discussed above, when  $\alpha > 0$ , the definition fo the ALAAM change statistic for GWActivity (5) means that the conditional log-odds of a node having the outcome attribute ( $y_i = 1$ )



Figure 3: Scatterplots of the effect on the mean degree of nodes that have the outcome attribute, of varying the Activity parameter (left) or Geometrically Weighted Activity parameter (right), for the GitHub social network (top) and undirected Pokec social network (bottom). The red dashed horizontal line shows the observed value.



Figure 4: Effect on the mean out-degree of nodes that have the outcome attribute, of varying the Sender parameter (left) or Geometrically Weighted Sender parameter (right), for the high school friendship network. The red dashed horizontal line shows the observed value. These plots show the mean over 100 samples for each value of the parameter.

is higher for a low degree node than a high degree node, and hence a positive value of the corresponding parameter will result in more low degree nodes having the outcome attribute than would otherwise be the case. Regrettably, this would seem likely to lead to confusion similar to that described by Levy et al. (2016): it seems counter-intuitive that a positive parameter should lead to a preference for the outcome attribute on low degree nodes (rather than high degree nodes).

Figure 3 shows the effect of the Activity and GWActivity parameters on the mean degree of nodes with the outcome attribute. (These are from the same simulations as those described for Figure 1 and Figure 2). It is evident that the mean degree of nodes with the outcome attribute does not have a simple relationship to the Activity parameter, first increasing, then after a near discontinuity, decreasing. In contrast, the mean degree of such nodes decreases smoothly as the GWActivity parameter is increased.

Figure 4 shows similar plots for the, much smaller, high school friendship network. Being a directed network, this plot shows the effect of the GWSender parameter on mean out-degree of nodes with the outcome attribute. For this small network, there is no near-discontinuity when using the Sender statistic (and in fact, an ALAAM for this network can be estimated with the Sender and Receiver parameters, as shown in Section 5.1). The pattern of the mean out-degree of nodes with  $y_i = 1$  increasing with the Sender parameter, and then decreasing, while the GWSender parameter results in a smooth decrease, is, however, again apparent.

The small size of the high school friendship network also makes it more practical to visualize the degree distributions in order to more closely examine the effect of the GWSender parameter. Figure 5 shows the effect of large magnitude negative and positive GWSender parameters on the distribution of the out-degree of nodes with the outcome attribute, compared with the distribution resulting from a random assignment of the outcome attribute to the nodes. The ALAAM models were simulated with the Density parameter logit(p) = -0.3930425, where p = 0.4029851 is the observed relative frequency of nodes with the outcome attribute, male gender. The random outcome vectors have each element one with probability  $\overline{\sum y^{(k)}/N}$  (where  $y^{(k)}$  is the *k*th ( $1 \le k \le 100$ ) ALAAM sample), so that the mean attribute density is the same as as that from the ALAAM simulations. For the negative GWSender parameter ( $\theta_{GWSender} = -15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ),  $\overline{\sum y^{(k)}/N} = 0.2191791$ , and for the positive GWSender parameter ( $\theta_{GWSender} = 15$ ).

For the negative GWSender parameter (Fig. 5(a)), the distribution is less skewed than for the positive GWSender parameter (Fig. 5(b)). The mean out-degree of nodes with the outcome attribute is higher than that for the random (and observed) outcomes for the negative parameter value, and lower than that for the random (and observed) outcomes for the positive parameter value. This reflects the interpretation discussed above (in the context of the undirected GWActivity parameter), that a positive GWSender parameter will lead to a tendency for the outcome attribute to be present on low (rather than high) out-degree nodes.



Figure 5: Effect of (a) negative, and (b) positive, GWSender [ $\alpha = \ln(2)$ ] parameters on the out-degree distribution of nodes with the outcome attribute (male gender) in the high school friendship network. The orange boxplots show the results for 100 outcome vectors simulated from the ALAAM models, and the purple boxplots 100 random outcome vectors where each element is 1 with probability  $\overline{\sum y/N}$ , so that the mean attribute density is the same as that of the outcome vectors simulated from the ALAAM. The solid green vertical line shows the observed mean out-degree of nodes with the outcome attribute. Similarly, the orange dashed vertical line is the mean for the ALAAM, and the purple dashed vertical line for the random outcome vectors.

# 5 Empirical examples of ALAAMs with the new parameters

### 5.1 Small network

Table 6 shows six ALAAM models for the high school friendship network, with male gender as the "outcome" binary variable. Table 7 shows the goodness-of fit-results for these models: in all cases, the t-ratio is less than 1.0 in magnitude, indicating a good fit for that statistic. Models 1–3 are relatively simple models, starting with Sender and Receiver and progressively adding EgoInTwoStar and EgoOutTwoStar (Model 2) and then also EgoInThreeStar and EgoOutThreeStar (Model 3). Model 4 is an equivalent model, but using GWSender and GWReceiver instead of the Sender, Receiver and in- and out-star effects. Model 5 adds a number of additional effects, including transitive triangles and homophily on school class, to the Sender/Receiver/star model (Model 3), while Model 6 adds the extra effects to the GWSender/GWReceiver model (Model 4).

The only parameter that is statistically significant across multiple models is Contagion, which is positive and significant in all cases (except Model 5, where it is not significant). This indicates homophily on (male) gender, consistent with ERGM models for (an undirected version of) this network (Stivala, 2020a; Kevork and Kauermann, 2021). (I estimated an ERGM model similar to that in Stivala (2020a), but for the original directed network, which finds a positive but non-significant effect for gender homophily; data not shown).

Table 6: Parameter estimates with standard errors for ALAAM estimated using ALAAMEE with the stochastic approximation algorithm for the SocioPatterns high school friendship network, with male gender as the outcome variable.

| Effect                           | Model 1 | Model 2 | Model 3                     | Model 4 | Model 5                      | Model 6                      |
|----------------------------------|---------|---------|-----------------------------|---------|------------------------------|------------------------------|
| Density                          | -0.648  | -0.180  | 0.527                       | -1.687  | 0.693                        | -2.637                       |
| Sender                           | -0.022  | -0.543  | -0.899                      | (0.357) | -0.857                       | (1.002)                      |
| EgoOutTwoStar                    | (0.097) | 0.087   | 0.214                       |         | 0.220                        | _                            |
| EgoOutThreeStar                  |         | (0.048) | -0.021                      |         | -0.018                       |                              |
| Receiver                         | -0.138  | 0.159   | (0.020)<br>0.088<br>(0.463) |         | (0.029)<br>-0.129<br>(0.654) |                              |
| EgoInTwoStar                     | (0.105) | -0.050  | -0.016                      | —       | 0.096                        | —                            |
| EgoInThreeStar                   | —       | (0.011) | -0.007                      | —       | -0.022                       |                              |
| GWSender [ $\alpha = \ln(2)$ ]   |         |         |                             | 3.565   |                              | 5.259                        |
| GWReceiver [ $\alpha = \ln(2)$ ] |         |         | —                           | -0.240  |                              | -0.578                       |
| Contagion                        | 0.239   | 0.258   | 0.253                       | 0.206   | 0.631                        | 0.725<br>(0.253)             |
| Reciprocity                      |         |         | (0.070)                     |         | -0.333                       | -0.092                       |
| Contagion Reciprocity            | —       |         | _                           |         | (0.003)<br>-0.785<br>(0.729) | (0.512)<br>-1.048<br>(0.585) |
| MixedTwoStarSink                 | —       | —       |                             | —       | 0.003<br>(0.036)             | -0.008                       |
| MixedTwoStarSource               |         |         |                             |         | 0.013<br>(0.041)             | 0.025                        |
| TransitiveTriangleT1             |         |         | —                           |         | -0.061                       | -0.048                       |
| TransitiveTriangleT3             |         |         | —                           |         | -0.010                       | -0.012                       |
| SenderMatch Class                |         |         | —                           | —       | -0.059                       | 0.105<br>(0.282)             |
| ReceiverMatch Class              |         |         | _                           |         | 0.005                        | -0.031                       |
| ReciprocityMatch Class           |         |         |                             |         | 0.319<br>(0.669)             | (0.323)<br>0.184<br>(0.498)  |

Parameter estimates that are statistically significant at the nominal p < 0.05 level are shown in bold.

Table 7: ALAAM goodness-of-fit t-ratios for the SocioPatterns high school social network ALAAM models (Table 6).

| Effect                           | Model 1 | Model 2 | Model 3 | Model 4 | Model 5 | Model 6 |
|----------------------------------|---------|---------|---------|---------|---------|---------|
| AlterInTwoStar2                  | 0.230   | 0.292   | 0.285   | 0.482   | 0.267   | 0.286   |
| AlterOutTwoStar2                 | 0.131   | 0.178   | 0.201   | 0.191   | 0.050   | 0.073   |
| Contagion                        | -0.009  | -0.016  | -0.015  | -0.043  | 0.006   | -0.018  |
| Contagion Reciprocity            | 0.454   | 0.518   | 0.519   | 0.519   | 0.028   | 0.002   |
| CyclicTriangleC1                 | 0.520   | 0.772   | 0.798   | 0.942   | 0.097   | 0.143   |
| CyclicTriangleC3                 | 0.610   | 0.772   | 0.797   | 0.830   | 0.193   | 0.182   |
| Density                          | 0.031   | -0.007  | -0.004  | 0.049   | -0.021  | -0.035  |
| EgoInThreeStar                   |         |         | -0.062  |         | 0.029   |         |
| EgoInTwoStar                     | -0.017  | -0.027  | -0.036  | 0.434   | 0.015   | 0.076   |
| EgoOutThreeStar                  |         |         | -0.034  |         | -0.082  |         |
| EgoOutTwoStar                    | -0.232  | -0.036  | -0.004  | -0.207  | -0.062  | -0.142  |
| GWReceiver [ $\alpha = \ln(2)$ ] |         |         |         | 0.146   |         | -0.048  |
| GWSender [ $\alpha = \ln(2)$ ]   |         |         |         | 0.166   |         | -0.088  |
| MixedTwoStar                     | 0.007   | 0.113   | 0.122   | 0.277   | 0.037   | 0.027   |
| MixedTwoStarSink                 | 0.136   | 0.192   | 0.194   | 0.486   | -0.003  | 0.024   |
| MixedTwoStarSource               | 0.141   | 0.201   | 0.232   | 0.198   | -0.033  | 0.008   |
| Receiver                         | 0.011   | -0.016  | -0.013  | 0.202   | -0.004  | 0.009   |
| ReceiverMatch Class              |         |         |         |         | -0.018  | 0.008   |
| Reciprocity                      | 0.410   | 0.433   | 0.434   | 0.525   | -0.013  | 0.007   |
| ReciprocityMatch Class           |         |         |         |         | -0.027  | 0.011   |
| Sender                           | 0.019   | -0.020  | 0.010   | -0.086  | -0.037  | -0.039  |
| SenderMatch Class                |         |         |         |         | -0.043  | -0.011  |
| TransitiveTriangleD1             | 0.290   | 0.546   | 0.581   | 0.508   | 0.049   | 0.055   |
| TransitiveTriangleT1             | 0.271   | 0.453   | 0.492   | 0.601   | -0.005  | 0.027   |
| TransitiveTriangleT3             | 0.310   | 0.438   | 0.460   | 0.481   | 0.037   | 0.007   |
| TransitiveTriangleU1             | 0.236   | 0.344   | 0.377   | 0.696   | -0.032  | 0.030   |

Although they are statistically non-significant, so we can make no inferences from them, it is instructive to compare the estimated Sender, EgoOutTwoStar, EgoOutThreeStar, Receiver, EgoInTwoStar, and EgoInThreeStar parameters in Model 5, with the GWSender and GWReceiver parameter estimates in Model 6 (Table 6). In Model 5, Sender is negative, EgoOutTwoStar is positive, and EgoOutThreeStar is negative; they have alternating signs, as discussed in Section 4.1. Receiver is negative, EgoInTwoStar positive, and EgoInThreeStar negative, so again the signs are alternating (note that Receiver and EgoInTwoStar have swapped signs relative to Model 3, however). In Model 6, GWSender is positive, while GWReceiver is negative. Figure 6 shows that Model 6 fits the in-degree and out-degree distributions of nodes with the outcome attribute well, although a simple random assignment of the outcome attribute with the same density is not much worse (which, given that the GWSender and GWReceiver parameters are not statistically significant, should not be surprising).

# 5.2 Large networks

Table 8 shows ALAAM parameters estimated for the GitHub network with developer type as the "outcome" binary attribute. I was unable to estimate a converged (non-degenerate) model for this data using the Density, Activity, and Contagion parameters, but using GWActivity instead the model is converged and non-degenerate, as shown in Figure 7, which shows trace plots and histograms of outcome vectors simulated from the model in Table 8, along with the observed values of the statistics corresponding to the parameters in the model. The observed values are central in the (approximately normal) distribution of the simulated values, indicating that the model is converged and not near-degenerate. The only parameter (other than Density) that is statistically significant in this model is GWActivity, which is positive. As discussed in Section 4.1, this means we expect that more low-degree nodes will have the outcome attribute than would otherwise be the case (conditional on all the other effects in the model, and on the degree distribution itself, since the network is fixed in the ALAAM). This is consistent with what we observe simply from the degrees of the nodes with and without the outcome attribute shown in Table 5; nodes with the outcome



Figure 6: Goodness-of-fit on in-degree (top) and out-degree (bottom) distributions of nodes with the outcome attribute (male gender) for ALAAM Model 6 (Table 6). The orange boxplots show the results for 100 outcome vectors simulated from the ALAAM, and the purple boxplots 100 random outcome vectors where each element is 1 with probability  $\overline{\sum y/N}$ , so that the mean attribute density is the same as that of the outcome vectors simulated from the ALAAM. The solid green vertical line shows the observed value. The the orange dashed vertical line is the mean for the ALAAM, and the purple dashed vertical line for the random outcome vectors.

attribute have lower mean degree than the overall mean degree.

Table 8: ALAAM estimated using ALAAMEE with the equilibrium expectation algorithm for the GitHub social network, with developer type as the outcome variable.

| Effect                           | Estimate | Std. error |   |
|----------------------------------|----------|------------|---|
| Density                          | -1.287   | 0.033      | * |
| GWActivity [ $\alpha = \ln(2)$ ] | 1.712    | 0.127      | * |
| Contagion                        | 0.002    | 0.001      |   |

Asterisks indicate statistical significance at the p < 0.05 level. Results from 100 parallel runs.

An ALAAM model for the undirected Pokec network with male gender as the "outcome" attribute is shown in Table 9, with the degeneracy check plots in Figure 8 showing that the model is converged. I was unable to estimate a converged (non-degenerate) model with this network when the Activity parameter was included, but using GWActivity instead solves this problem. All the parameters in this model are statistically significant. The negative Contagion parameter indicates heterophily on (male) gender, while the positive Age parameter indicates that males are likely to be older than females. This is consistent with simple descriptive statistics for this data: assortativity (Newman, 2003) on the "male" binary attribute is negative (r = -0.0053), and the mean age for male actors (25.1) is higher than that for non-male actors (23.84) with the difference significant according to Welch's *t*-test (p < 0.0001). The positive GWActivity parameter indicates, as discussed in Section 4.1, that low degree nodes are more likely to represent male actors than would otherwise be the case. This is as we might expect, given that male (outcome  $y_i = 1$ ) nodes have lower mean degree than the mean degree than others (Table 5).

Table 9: ALAAM estimated using ALAAMEE with the equilibrium expectation algorithm for the undirected Pokec online social network, with male gender as the outcome variable.

| Effect                           | Estimate | Std. error |   |
|----------------------------------|----------|------------|---|
| Density                          | -0.188   | < 0.001    | * |
| GWActivity [ $\alpha = \ln(2)$ ] | 0.077    | 0.001      | * |
| Contagion                        | -0.005   | < 0.001    | * |
| Age                              | 0.009    | < 0.001    | * |

Asterisks indicate statistical significance at the p < 0.05 level. Results from 100 parallel runs.

A more complex ALAAM model for the directed Pokec network with male gender as "outcome" variable, is shown in Table 10. I could not find a converged (non-degenerate) ALAAM model for this network using the Sender and Receiver parameters, but as shown in Figure 9, this model using GWSender and GWReceiver converges well. Again, all the parameters in this model are statistically significant. As we expect given the results for the undirected network, the Age effect is positive and the Contagion effect negative; this is also consistent with the ERGM model of this network in Stivala et al. (2020a). However Contagion Reciprocity is positive, indicating that actors connected by a reciprocated (mutual) tie are more likely to both be male (given the other effects in the model, including specifically the negative Contagion parameter, indicating that a male actor on both ends of a tie is under-represented). The GWSender and GWReceiver parameters are of different signs: GWSender is negative, and GWReceiver positive. Again, as per the discussion Section 4.1, this is as we expect, given that male actors have higher mean out-degree, but lower mean in-degree than others (Table 5).

### 6 Conclusions and future work

I have shown that the problem of near-degeneracy can occur in simple ALAAMs applied to empirical networks, preventing the estimation of such models in some examples. I defined the geometrically weighted activity, geometrically weighted sender, and geometrically weighted receiver statistics, analogous to the geometrically weighted degree statistics for ERGMs described by Snijders et al. (2006), and showed that they avoid this problem, and allow ALAAM parameters to be estimated for these networks. I described the interpretation of these new parameters, with illustrative examples.

Table 10: ALAAM estimated using ALAAMEE with the equilibrium expectation algorithm for the directed Pokec online social network, with male gender as the outcome variable.

| Effect                           | Estimate | Std. error |   |
|----------------------------------|----------|------------|---|
| Density                          | -0.015   | 0.002      | * |
| GWSender [ $\alpha = \ln(2)$ ]   | -0.509   | 0.011      | * |
| GWReceiver [ $\alpha = \ln(2)$ ] | 0.517    | 0.011      | * |
| Reciprocity                      | 0.023    | < 0.001    | * |
| Contagion                        | -0.028   | < 0.001    | * |
| Contagion Reciprocity            | 0.019    | 0.001      | * |
| Age                              | 0.008    | < 0.001    | * |
|                                  |          |            |   |

Asterisks indicate statistical significance at the p < 0.05 level. Results from 100 parallel runs.



Figure 7: Degeneracy check for the GitHub social network ALAAM (Table 8). Trace plots and histograms show statistics of 100 outcome vectors simulated from the model. The blue lines on the histograms show mean and 95% confidence interval, and red lines show the observed values.



Figure 8: Degeneracy check for the undirected Pokec social network ALAAM (Table 9). Trace plots and histograms show statistics of 100 outcome vectors simulated from the model. The blue lines on the histograms show mean and 95% confidence interval, and red lines show the observed values.



Figure 9: Degeneracy check for the directed Pokec social network ALAAM (Table 10). Trace plots and histograms show statistics of 100 outcome vectors simulated from the model. The blue lines on the histograms show mean and 95% confidence interval, and red lines show the observed values.

In this work, I defined these statistics and demonstrated the use for one-mode undirected and directed networks. A simple extension would be to two-mode (bipartite) networks, which might allow a converged ALAAM to be found for the larger director interlock network (Evtushenko and Gastner, 2020) which I was unable to find, while I could find a converged ALAAM for the smaller director interlock network in Stivala et al. (2023b),

In the examples shown here, I found that only the geometrically weighted activity (or sender and receiver) statistic was necessary to overcome the problem of near-degeneracy: the Contagion statistic, when used with geometrically weighted activity (or sender and receiver) statistics, did not seem to be problematic. Indeed, when I experimented with a "geometrically weighted contagion" statistic, I found it to be not just unnecessary, but actually deleterious to model convergence. Given that I used only simple models for the large network examples, this leaves open the question of whether or not geometrically weighted statistics are necessary or useful for triangular configurations in the ALAAM (as they are in ERGM).

Some problems remain, however. As discussed in Section 4.1, interpretation of the new parameters is likely to be confusing, given the counter-intuitive meaning of a positive parameter indicating a propensity for the outcome attribute to be present on low (rather than high) degree nodes. Simulation experiments such as those shown in Figure 5, which, not coincidentally, somewhat resembles the output of the interactive R application created to help with the interpretation of the statnet gwdegree parameter (Levy, 2016), could help with this. However the interpretation is (aside from the potential for the sign-based confusion), inherently difficult, as it is linked to the degree distribution of nodes with the outcome attribute, and conditional not only on all the other parameters in the model, but also on the degree distribution of the network itself (which is fixed in the ALAAM). This is particularly complicated in the case of directed networks, in which there is both an in-degree and out-degree distribution, and interpretation of the GWS ender and GWReceiver parameters are conditional on each other. In this work I have described the interpretation of these parameters as illustrative examples, however in empirical applications it might be advisable to refrain from making substantive claims based on these parameters, and just consider them as "controls" for the degree distribution of nodes with the outcome attribute, needed for correct interpretation of the Contagion (and other) parameters. Of course, this is assuming that parameter interpretation is actually want we want do — and perhaps it is not, and we would rather use the model to generate simulations in order test predictions regarding their inability to fit some statistic (Martin, 2020), or to experiment with simulations from different models with slightly modified parameters (Steglich and Snijders, 2022).

Another avenue for future work is that a value for the decay parameter  $\alpha$  has to be specified. The default value of  $\alpha = \ln(2)$  appears to work well on the examples in this work, but it may have to be adjusted for better convergence or model fit on other networks, which would involve a process of trial and error, or, more systematically, "grid search" as, for example, done for the analogous  $\lambda$  parameter in ERGMs in Stivala and Lomi (2021). Estimating this parameter would make the model a "curved ALAAM", which cannot be estimated by the methods used in this work.

In this work, I overcame the problem of near-degeneracy in ALAAMs by defining a geometrically weighted activity statistic, analogous to the most frequently used technique of avoiding the problem in ERGMs. There are, however, other ways of avoiding this problem in ERGMs, which could potentially be applied to ALAAMs. These included the "tapering" method (Fellows and Handcock, 2017; Blackburn and Handcock, 2023), and the "degeneracy-restricted" method (Karwa et al., 2022), as well as other forms of additional structure discussed in Schweinberger et al. (2020) such as multilevel, block and spatial structure. An alternative approach might be to consider an ALAAM analogue of the latent order logistic (LOLOG) model (Fellows, 2018; Clark and Handcock, 2022).

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# Data availability statement

The SocioPatterns high school friendship data (Mastrandrea et al., 2015) is available from http://www.sociopatterns. org/datasets/high-school-contact-and-friendship-networks/. The "Pokec" (Takac and Zabovsky, 2012) data is available from the Stanford large network dataset collection (Leskovec and Krevl, 2014) at http:// snap.stanford.edu/data/soc-Pokec.html. The "GitHub" (Rozemberczki et al., 2021) online social network data is available from the same collection at http://snap.stanford.edu/data/github-social. html. All other data, source code, and scripts are freely available from https://github.com/stivalaa/ ALAAMEE.

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